Reconstruction of Quantum Illumination with Gaussian States

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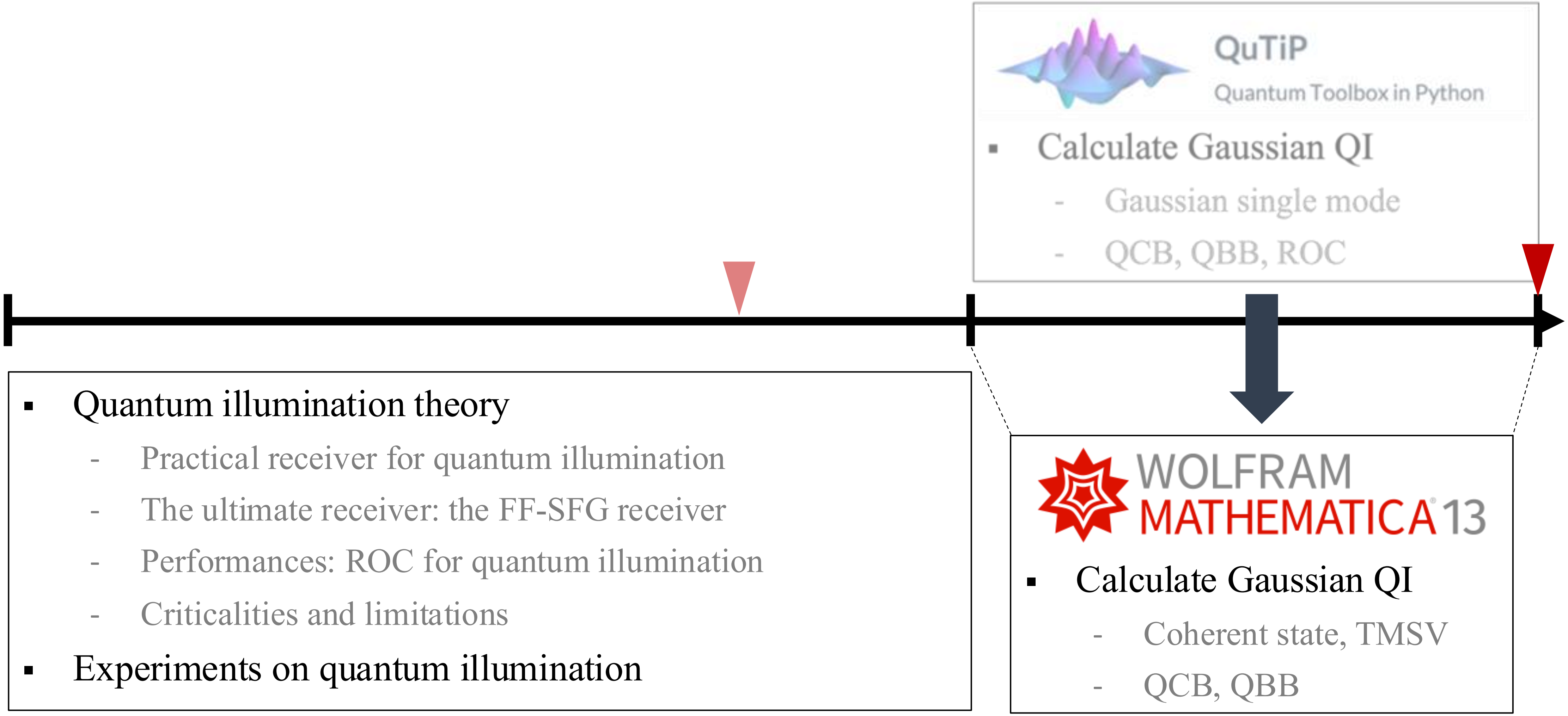
Hyeonjae Kim

Final Presentation for Exploration Project in Physics 2 (PHYS0352)

Contents

Status



- Quantum illumination theory
  - Practical receiver for quantum illumination
  - The ultimate receiver: the FF-SFG receiver
  - Performances: ROC for quantum illumination
  - Criticalities and limitations
- Experiments on quantum illumination

QuTiP
Quantum Toolbox in Python

- Calculate Gaussian QI
  - Gaussian single mode
  - QCB, QBB, ROC

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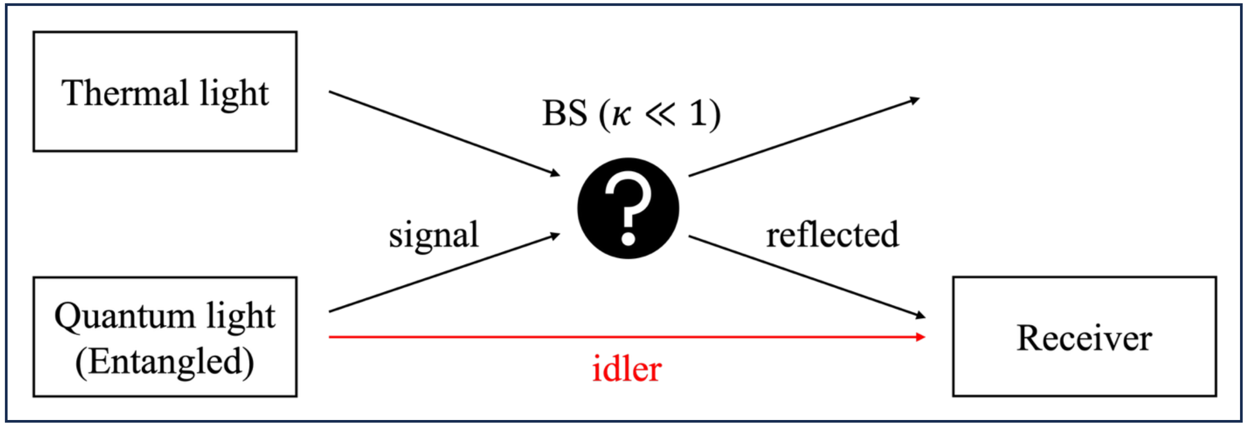
- Calculate Gaussian QI
  - Coherent state, TMSV
  - QCB, QBB

Recall : Quantum illumination

- Quantum illumination

: Quantum target detection protocol



case	decision	real	
1	H_0	H_0	correct
2	H_1	H_0	false alarm error
3	H_1	H_1	correct
4	H_0	H_1	miss error

Choose E_i $(i = 0,1)$ that minimize $P_e = \omega_0 \mathrm{tr}(E_1 \rho_0) + \omega_0 \mathrm{tr}(E_1 \rho_1)$

$$P_e = \frac{1 - tr\left|\omega_1 {\rho_1}^{\otimes M} - \omega_0 {\rho_0}^{\otimes M}\right|}{2}$$: Helstrom bound

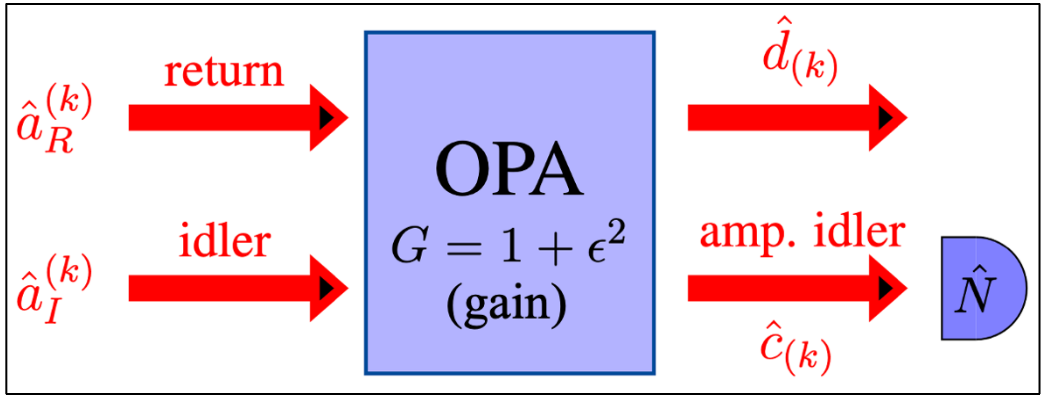
$$P_e \leq \frac{1}{2} e^{-M\xi_{QC}} \leq \frac{1}{2} e^{-M\xi_{QB}}$$

$\xi_{QB} = -\log[\mathrm{tr}(\sqrt{\rho_0}\sqrt{\rho_1})]$: Quantum Bhattachayya error exponent

$\xi_{QC} = -\log[\min_{0\leq\alpha\leq1} \mathrm{tr}({\rho_0}^{\alpha} {\rho_1}^{1-\alpha})]$: Quantum Chernoff error exponent

Practical receiver

- OPA(optical parametric amplifier)



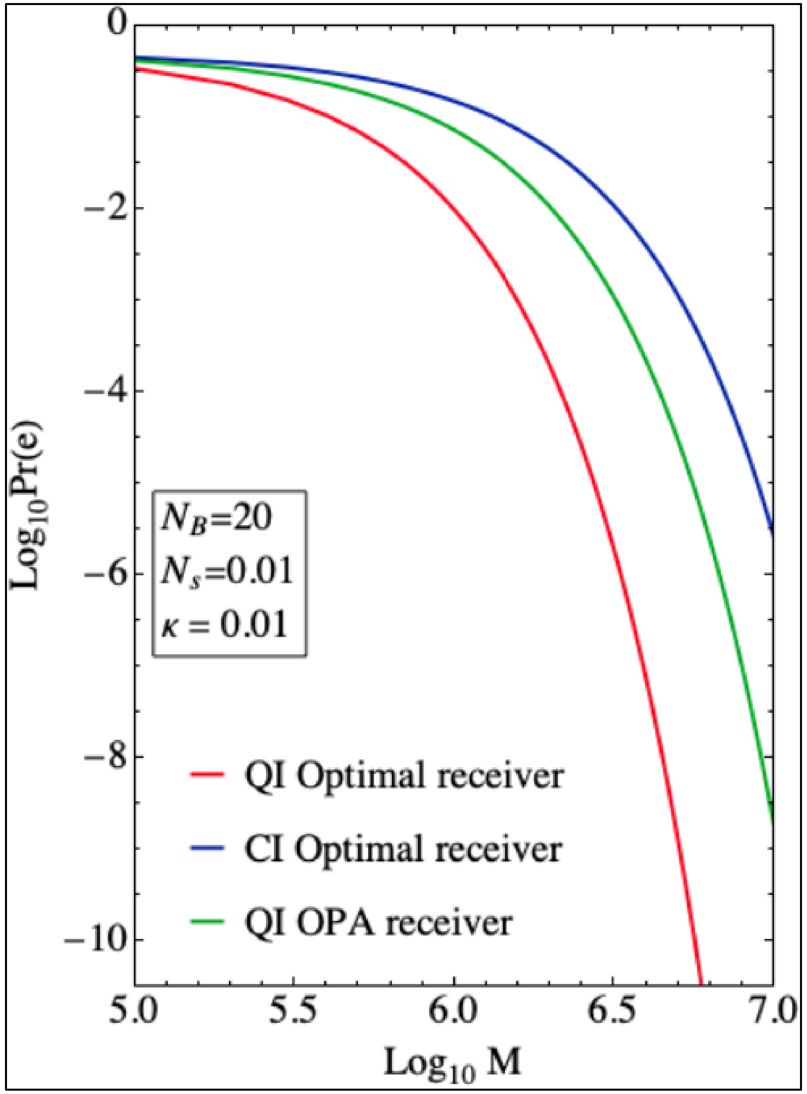
$$\hat{c}_{(k)} = \sqrt{G}\hat{a}_I{}^{(k)} + \sqrt{G-1}\hat{a}_R{}^{\dagger(k)} \qquad \hat{d}_{(k)} = \sqrt{G}\hat{a}_R{}^{(k)} + \sqrt{G-1}\hat{a}_I{}^{\dagger(k)}$$

$$N_k = \langle \hat{c}_{(k)}{}^{\dagger}\hat{c}_{(k)} \rangle$$

$$N_0 = GN_S + (G-1)(1+N_B)$$

$$N_1 = GN_S + (G-1)(1+N_B+\kappa N_S) + 2\sqrt{G(G-1)}\sqrt{\kappa N_S(N_S+1)}$$

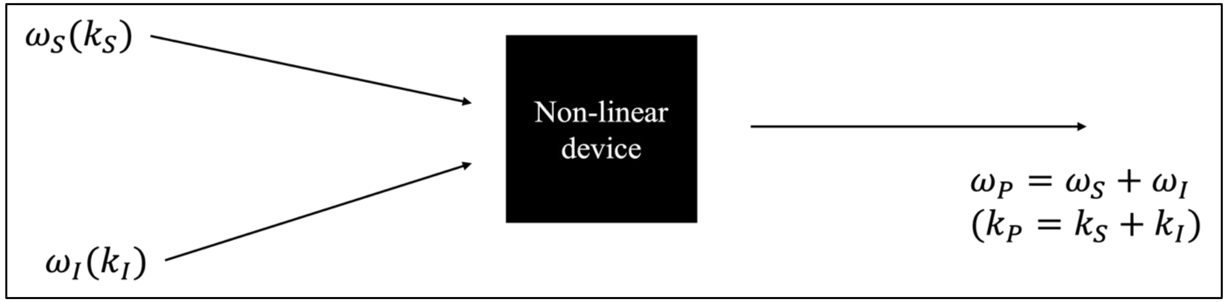
$$\boxed{P_{e,OPA} \le \frac{1}{2} Q_B{}^{M}} \qquad Q_B = \frac{1}{\sqrt{(1+N_1)(1+N_0)} - \sqrt{N_0 N_1}}$$



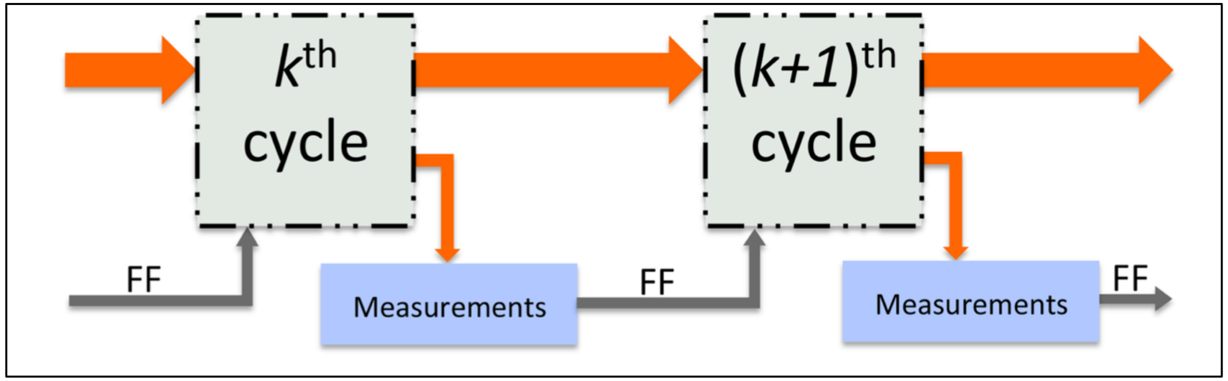
Practical receiver

- FF-SFG receiver

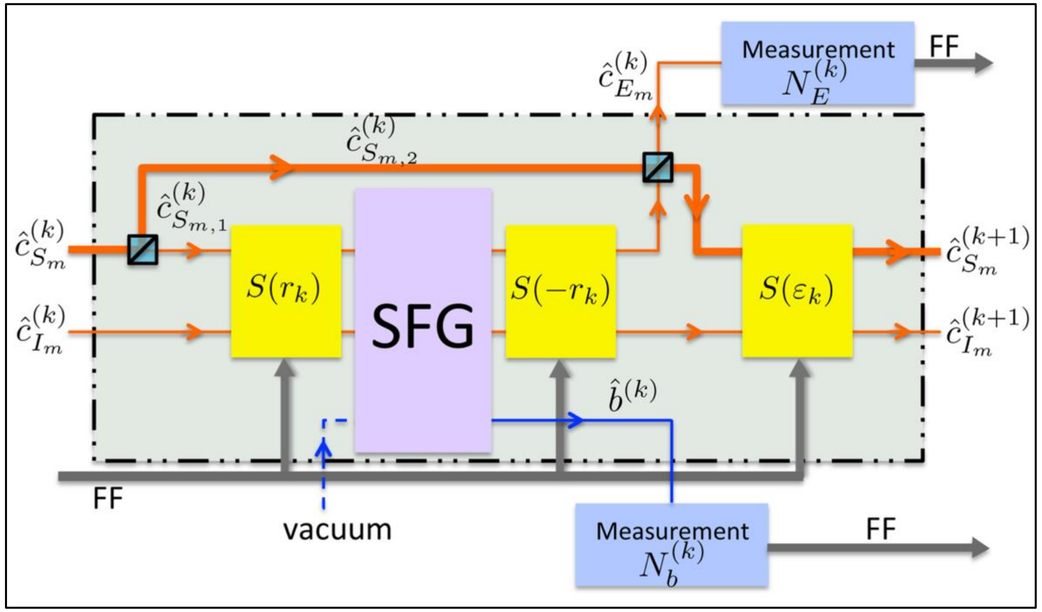
SFG(sum frequency generation)



FF(feed-forward) circuit



FF-SFG(k-th cycle)



- Saturate the Chernoff bound
- Unit efficiency at single pair level
- $|0\rangle$ (H_0 is true), $|N_S \kappa M / N_B\rangle$ (H_1 is true)

Performance : ROC

- ROC for quantum illumination

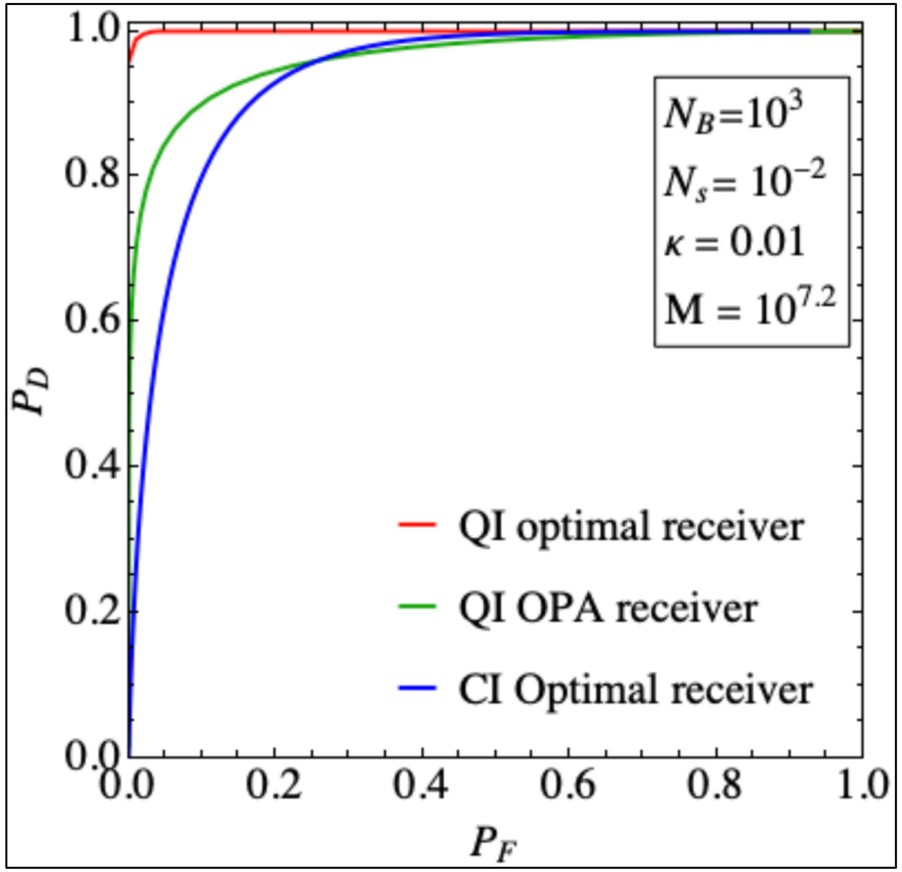
Neyman-Pearson hypothesis testing

$P_F = \mathrm{tr}(E_1\rho_0) = \alpha$ (fixed) $P_M = \mathrm{tr}(E_0\rho_1)$

$\Rightarrow P_D = 1 - P_M = \mathrm{tr}((\mathbb{I} - E_0)\rho_1) = \mathrm{tr}(E_1\rho_1)$

$\rho_0 = |\psi_0\rangle\langle\psi_0|$ $\rho_1 = |\psi_1\rangle\langle\psi_1|$ (pure states)

$$P_D = \begin{cases} \left(\sqrt{P_F(1-h)} + \sqrt{(1-P_F)h}\right)^2 & 0 \le P_F \le 1-h \\ 1 & 1-h \le P_F \le 1 \end{cases}$$



- QI with the FF-SFG receiver

$$P_D = \begin{cases} \left(\sqrt{P_F(1-h)} + \sqrt{(1-P_F)h}\right)^2 & 0 \le P_F \le 1-h \\ 1 & 1-h \le P_F \le 1 \end{cases}$$

$h = 1 - \exp(-N_s\kappa M/N_B)$

Normal mode decomposition of Gaussian states

- Normal mode decomposition of Gaussian states $(\bar{\mathbf{x}}, \mathbf{S}, \{\nu_k\})$

Every Covariance matrix **V** exists a symplectic matrix **S**

$$\mathbf{V} = \mathbf{S}\underbrace{[\bigoplus_{k=1}^{n} \nu_k \mathbf{I}_k]}_{\begin{pmatrix} \nu_1 & & & & \\ & \nu_1 & & & \\ & & \ddots & & \\ & & & \nu_n & \\ & & & & \nu_n \end{pmatrix}}\mathbf{S}^T \quad \longleftrightarrow \quad \rho = \hat{U}_{\bar{\mathbf{x}},\mathbf{S}}\, [\bigotimes_{k=1}^{n} \sigma(\nu_k)]\hat{U}^{\dagger}{}_{\bar{\mathbf{x}},\mathbf{S}}$$

: Normal mode decomposition of the Gaussian state

$$\sigma(\nu_k) = \frac{2}{\nu_k + 1}\sum_{j=0}^{n}\left(\frac{\nu_k - 1}{\nu_k + 1}\right)^{j} |j\rangle_k\langle j|$$

: Thermal state

Two basic functions

$$\Lambda_s(x) = \frac{(x+1)^s + (x-1)^s}{(x+1)^s - (x-1)^s}$$

$$G_s(x) = \frac{2^s}{(x+1)^s - (x-1)^s}$$

(symplectic diagonalization)

$\{\nu_1, \cdots, \nu_n\}$: sympletic spectrum

Normal mode decomposition of Gaussian states

- Consider two arbitrary n-mode Gaussian state ρ_A and ρ_B with a normal mode decompositions $(\bar{\mathbf{x}}_\mathrm{A}, \mathbf{S}_\mathrm{A}, \{\alpha_k\})$ and $(\bar{\mathbf{x}}_\mathrm{B}, \mathbf{S}_\mathrm{B}, \{\beta_k\})$

$$\mathbf{d} = \bar{\mathbf{x}}_\mathrm{A} - \bar{\mathbf{x}}_\mathrm{B}$$

$$Q_s = \mathrm{Tr}({\rho_A}^s {\rho_B}^{1-s}) \quad \leftrightarrow \quad Q_s = \overline{Q}_s \exp\{-\frac{1}{2}\mathbf{d}^T[\mathbf{V}_\mathrm{A}(s) + \mathbf{V}_\mathrm{B}(1-s)]^{-1}\mathbf{d}\}$$

$$\overline{Q}_s = 2^n \prod_{k=1}^{n} G_s(\alpha_k) G_{1-s}(\beta_k) \Big/ \sqrt{\det[\mathbf{V}_\mathrm{A}(s) + \mathbf{V}_\mathrm{B}(1-s)]}$$

$$\mathbf{V}_\mathrm{A}(s) = \mathbf{S}_\mathrm{A}[\oplus_{k=1}^{n} \Lambda_s(\alpha_k)\mathbf{I}_k](\mathbf{S}_\mathrm{A})^T$$

$$\mathbf{V}_\mathrm{B}(1-s) = \mathbf{S}_\mathrm{B}[\oplus_{k=1}^{n} \Lambda_{1-s}(\beta_k)\mathbf{I}_k](\mathbf{S}_\mathrm{B})^T$$

$$P^{(N)}{}_{QC} = \frac{1}{2}[\inf_{0\le s\le 1} Q_s]^{(N)}$$: Chernoff bound

$$\boxed{P^{(N)}{}_{QB} = \frac{1}{2}[Q_{s=1/2}]^{(N)}}$$: Bhattacharyya bound

Reconstruction of Classical illumination

- Calculate Bhattacharyya bound (for the Coherent-state transmitter)

Mean vector

$$\bar{\mathbf{x}}_0 = (0\,,0) \quad \bar{\mathbf{x}}_1 = (\sqrt{\kappa N_S}\,,0)$$

Symplectic matrix

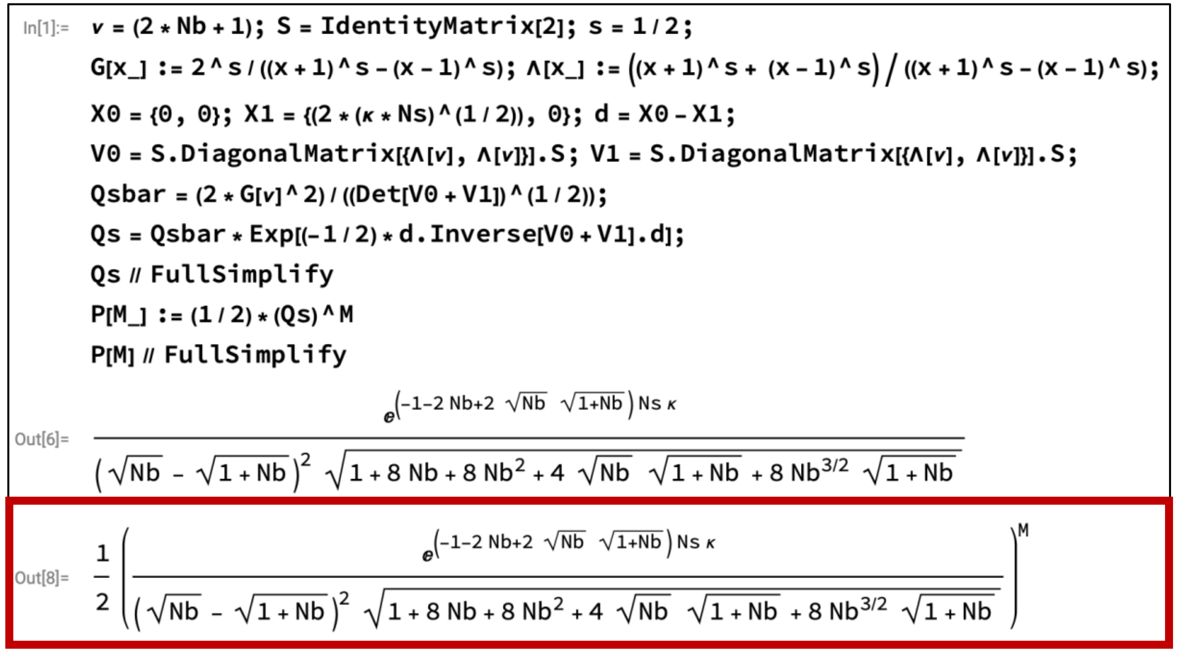
$$\mathbf{S_0} = \mathbf{S_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Symplectic spectra (eigenvalue)

$$\nu_1 = B = 2N_B + 1$$

Covariance matrix

$$V_0 = V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda(\nu_1) & 0 \\ 0 & \Lambda(\nu_1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



```
In[1]:= ν = (2 * Nb + 1); S = IdentityMatrix[2]; s = 1 / 2;
        G[x_] := 2 ^ s / ((x + 1) ^ s - (x - 1) ^ s); Λ[x_] := ((x + 1) ^ s + (x - 1) ^ s) / ((x + 1) ^ s - (x - 1) ^ s);
        X0 = {0, 0}; X1 = {(2 * (κ * Ns) ^ (1 / 2)), 0}; d = X0 - X1;
        V0 = S.DiagonalMatrix[{Λ[ν], Λ[ν]}].S; V1 = S.DiagonalMatrix[{Λ[ν], Λ[ν]}].S;
        Qsbar = (2 * G[ν] ^ 2) / ((Det[V0 + V1]) ^ (1 / 2));
        Qs = Qsbar * Exp[(-1 / 2) * d.Inverse[V0 + V1].d];
        Qs // FullSimplify
        P[M_] := (1 / 2) * (Qs) ^ M
        P[M] // FullSimplify
```

Out[6]=

$$\frac{e^{\left(-1-2\,\mathrm{Nb}+2\,\sqrt{\mathrm{Nb}}\,\sqrt{1+\mathrm{Nb}}\right)\mathrm{Ns}\,\kappa}}{\left(\sqrt{\mathrm{Nb}}-\sqrt{1+\mathrm{Nb}}\right)^2\sqrt{1+8\,\mathrm{Nb}+8\,\mathrm{Nb}^2+4\,\sqrt{\mathrm{Nb}}\,\sqrt{1+\mathrm{Nb}}+8\,\mathrm{Nb}^{3/2}\,\sqrt{1+\mathrm{Nb}}}}$$

Out[8]=

$$\frac{1}{2}\left(\frac{e^{\left(-1-2\,\mathrm{Nb}+2\,\sqrt{\mathrm{Nb}}\,\sqrt{1+\mathrm{Nb}}\right)\mathrm{Ns}\,\kappa}}{\left(\sqrt{\mathrm{Nb}}-\sqrt{1+\mathrm{Nb}}\right)^2\sqrt{1+8\,\mathrm{Nb}+8\,\mathrm{Nb}^2+4\,\sqrt{\mathrm{Nb}}\,\sqrt{1+\mathrm{Nb}}+8\,\mathrm{Nb}^{3/2}\,\sqrt{1+\mathrm{Nb}}}}\right)^M$$

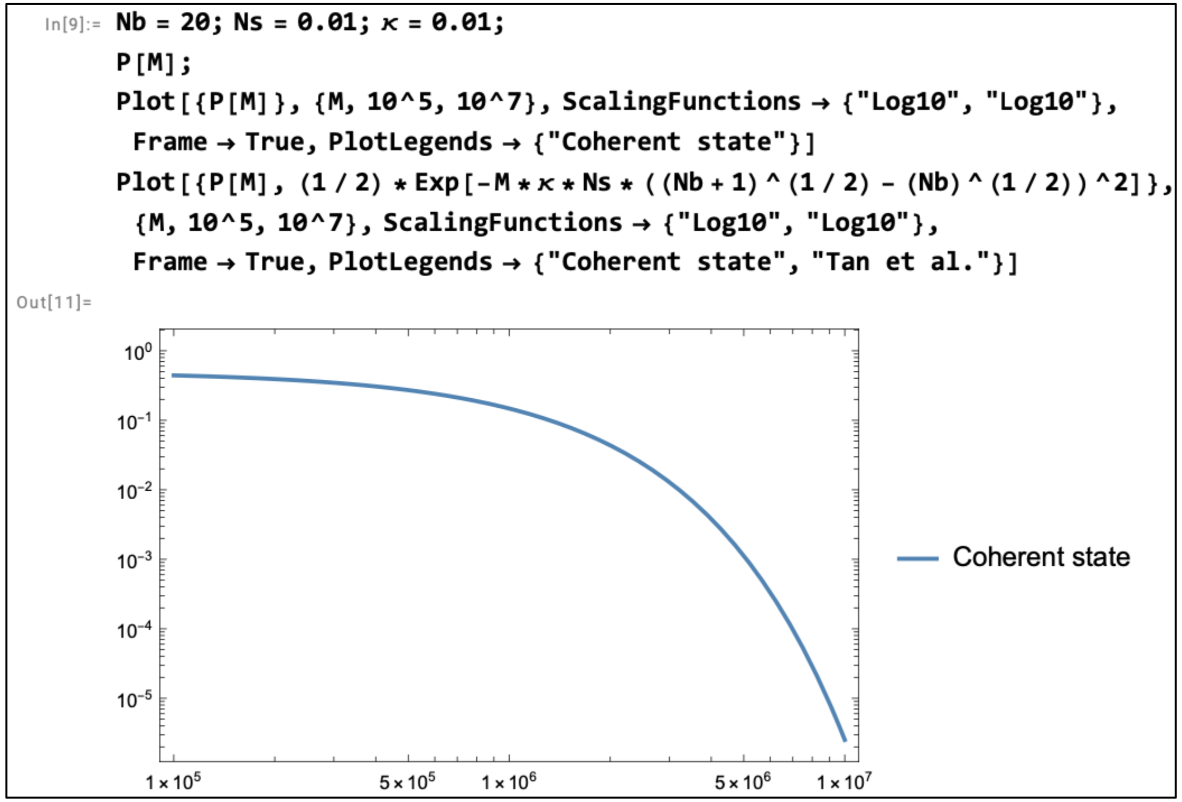
Analytic expression for Bhattacharyya bound

$$\Pr(e)_{\mathrm{CS}} \leq e^{-M\kappa N_S(\sqrt{N_B+1}-\sqrt{N_B})^2}/2$$

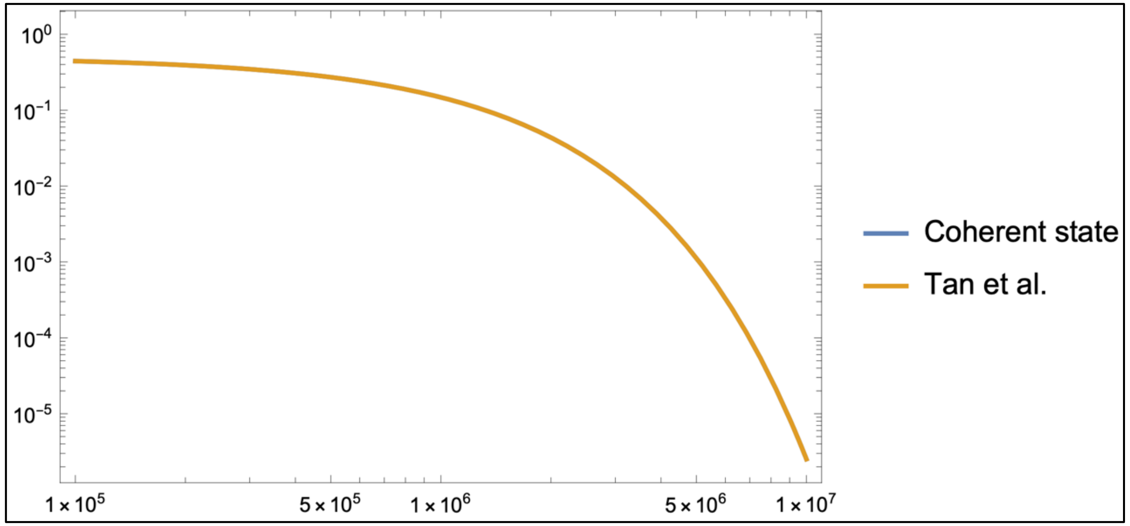
Reconstruction of Classical illumination

- Calculate Bhattacharyya bound (for the Coherent-state transmitter)

Upper bounds on the target-detection error probabilities for coherent state



```
In[9]:= Nb = 20; Ns = 0.01; κ = 0.01;
        P[M];
        Plot[{P[M]}, {M, 10^5, 10^7}, ScalingFunctions → {"Log10", "Log10"},
         Frame → True, PlotLegends → {"Coherent state"}]
        Plot[{P[M], (1 / 2) * Exp[-M * κ * Ns * ((Nb + 1) ^ (1 / 2) - (Nb) ^ (1 / 2)) ^ 2]},
         {M, 10^5, 10^7}, ScalingFunctions → {"Log10", "Log10"},
         Frame → True, PlotLegends → {"Coherent state", "Tan et al."}]
Out[11]=
```



- $N_B = 20, \kappa = 0.01, N_s = 0.01$
- Coincide with the Chernoff bounds

Reconstruction of Quantum illumination

- Calculate Bhattacharyya bound (for the TMSV transmitter)

Mean vector $\bar{\mathbf{x}}_0 = \bar{\mathbf{x}}_1 = (0,0,0,0)$

Symplectic matrix

$$\mathbf{S_0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{S_1} = \begin{pmatrix} \mathbf{X}_+ & \mathbf{X}_- \\ \mathbf{X}_- & \mathbf{X}_+ \end{pmatrix}$$

$$\mathbf{X}_\pm = \begin{pmatrix} x_\pm & 0 \\ 0 & x_\pm \end{pmatrix} \qquad x_\pm = \sqrt{\frac{A + S \pm \sqrt{(A+S)^2 - 4\kappa {C_q}^2}}{2\sqrt{(A+S)^2 - 4\kappa {C_q}^2}}}$$

Symplectic spectra (eigenvalue)

$$\nu_{0_1} = B = 2N_B + 1,\ \nu_{0_2} = S = 2N_S + 1$$

$$\nu_{1_k} = \frac{1}{2}\left[(-1)^k(S - A) + \sqrt{(A+S)^2 - 4\kappa {C_q}^2}\right] \quad (k = 1, 2)$$

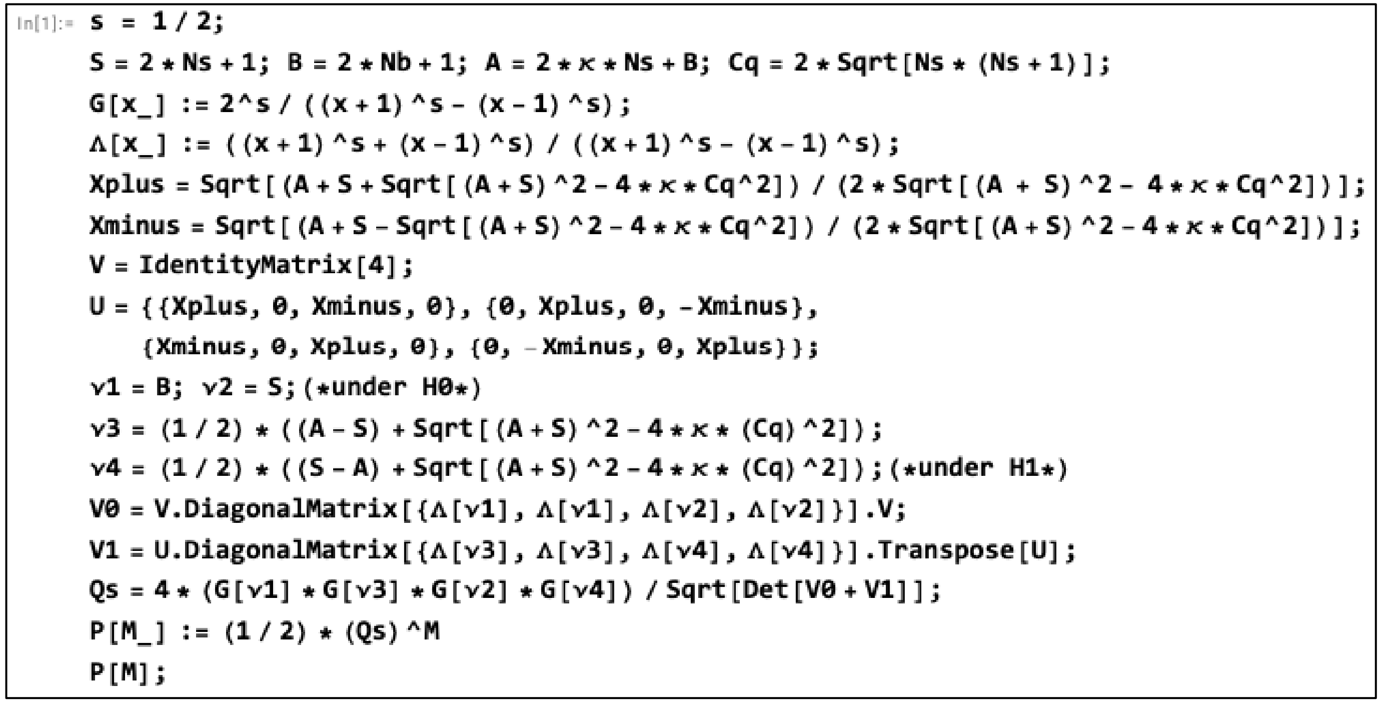
Covariance matrix

$$V_0 = \mathbf{S_0} \begin{pmatrix} \Lambda(\nu_{0_1}) & 0 & 0 & 0 \\ 0 & \Lambda(\nu_{0_1}) & 0 & 0 \\ 0 & 0 & \Lambda(\nu_{0_2}) & 0 \\ 0 & 0 & 0 & \Lambda(\nu_{0_2}) \end{pmatrix} \mathbf{S_0}^T$$

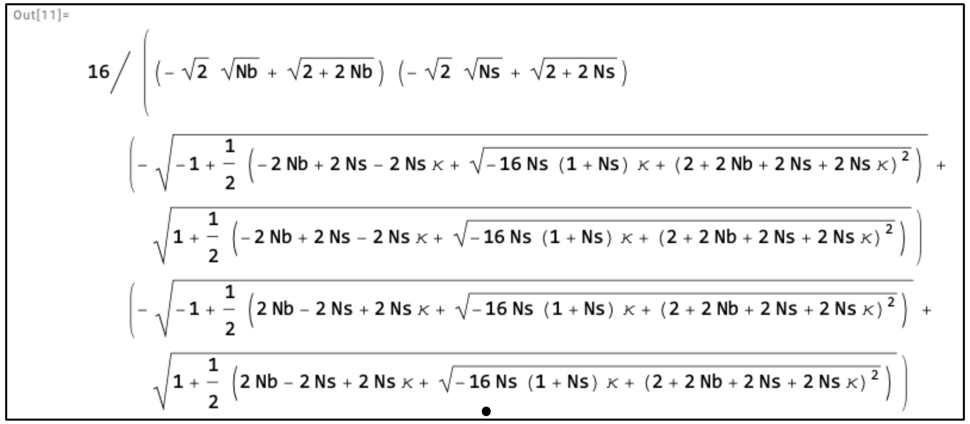
$$V_1 = \mathbf{S_1} \begin{pmatrix} \Lambda(\nu_{1_1}) & 0 & 0 & 0 \\ 0 & \Lambda(\nu_{1_1}) & 0 & 0 \\ 0 & 0 & \Lambda(\nu_{1_2}) & 0 \\ 0 & 0 & 0 & \Lambda(\nu_{1_2}) \end{pmatrix} \mathbf{S_1}^T$$

Reconstruction of Quantum illumination

- Calculate Bhattacharyya bound (for the TMSV transmitter)



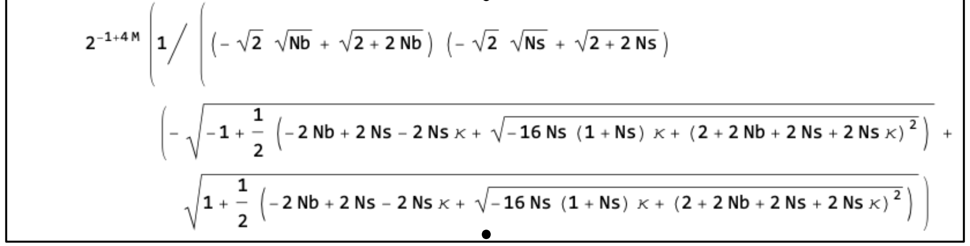
```
In[1]:= s = 1 / 2;
        S = 2 * Ns + 1; B = 2 * Nb + 1; A = 2 * κ * Ns + B; Cq = 2 * Sqrt[Ns * (Ns + 1)];
        G[x_] := 2^s / ((x + 1) ^s - (x - 1) ^s);
        Λ[x_] := ((x + 1) ^s + (x - 1) ^s) / ((x + 1) ^s - (x - 1) ^s);
        Xplus = Sqrt[(A + S + Sqrt[(A + S) ^2 - 4 * κ * Cq^2]) / (2 * Sqrt[(A + S) ^2 - 4 * κ * Cq^2])];
        Xminus = Sqrt[(A + S - Sqrt[(A + S) ^2 - 4 * κ * Cq^2]) / (2 * Sqrt[(A + S) ^2 - 4 * κ * Cq^2])];
        V = IdentityMatrix[4];
        U = {{Xplus, 0, Xminus, 0}, {0, Xplus, 0, -Xminus},
            {Xminus, 0, Xplus, 0}, {0, -Xminus, 0, Xplus}};
        ν1 = B; ν2 = S;(*under H0*)
        ν3 = (1 / 2) * ((A - S) + Sqrt[(A + S) ^2 - 4 * κ * (Cq) ^2]);
        ν4 = (1 / 2) * ((S - A) + Sqrt[(A + S) ^2 - 4 * κ * (Cq) ^2]);(*under H1*)
        V0 = V.DiagonalMatrix[{Λ[ν1], Λ[ν1], Λ[ν2], Λ[ν2]}].V;
        V1 = U.DiagonalMatrix[{Λ[ν3], Λ[ν3], Λ[ν4], Λ[ν4]}].Transpose[U];
        Qs = 4 * (G[ν1] * G[ν3] * G[ν2] * G[ν4]) / Sqrt[Det[V0 + V1]];
        P[M_] := (1 / 2) * (Qs) ^M
        P[M];
```



Out[11]=

$$\begin{aligned}&16\Big/\Big(\left(-\sqrt{2}\,\sqrt{Nb}+\sqrt{2+2\,Nb}\right)\left(-\sqrt{2}\,\sqrt{Ns}+\sqrt{2+2\,Ns}\right)\\&\quad\left(-\sqrt{-1+\frac{1}{2}\left(-2\,Nb+2\,Ns-2\,Ns\,\kappa+\sqrt{-16\,Ns\,(1+Ns)\,\kappa+(2+2\,Nb+2\,Ns+2\,Ns\,\kappa)^2}\right)}+\right.\\&\quad\left.\sqrt{1+\frac{1}{2}\left(-2\,Nb+2\,Ns-2\,Ns\,\kappa+\sqrt{-16\,Ns\,(1+Ns)\,\kappa+(2+2\,Nb+2\,Ns+2\,Ns\,\kappa)^2}\right)}\right)\\&\quad\left(-\sqrt{-1+\frac{1}{2}\left(2\,Nb-2\,Ns+2\,Ns\,\kappa+\sqrt{-16\,Ns\,(1+Ns)\,\kappa+(2+2\,Nb+2\,Ns+2\,Ns\,\kappa)^2}\right)}+\right.\\&\quad\left.\sqrt{1+\frac{1}{2}\left(2\,Nb-2\,Ns+2\,Ns\,\kappa+\sqrt{-16\,Ns\,(1+Ns)\,\kappa+(2+2\,Nb+2\,Ns+2\,Ns\,\kappa)^2}\right)}\right)\end{aligned}$$

⋮



Out[13]=

$$\begin{aligned}&2^{-1+4\,M}\Big(1\Big/\Big(\left(-\sqrt{2}\,\sqrt{Nb}+\sqrt{2+2\,Nb}\right)\left(-\sqrt{2}\,\sqrt{Ns}+\sqrt{2+2\,Ns}\right)\\&\quad\left(-\sqrt{-1+\frac{1}{2}\left(-2\,Nb+2\,Ns-2\,Ns\,\kappa+\sqrt{-16\,Ns\,(1+Ns)\,\kappa+(2+2\,Nb+2\,Ns+2\,Ns\,\kappa)^2}\right)}+\right.\\&\quad\left.\sqrt{1+\frac{1}{2}\left(-2\,Nb+2\,Ns-2\,Ns\,\kappa+\sqrt{-16\,Ns\,(1+Ns)\,\kappa+(2+2\,Nb+2\,Ns+2\,Ns\,\kappa)^2}\right)}\right)\end{aligned}$$

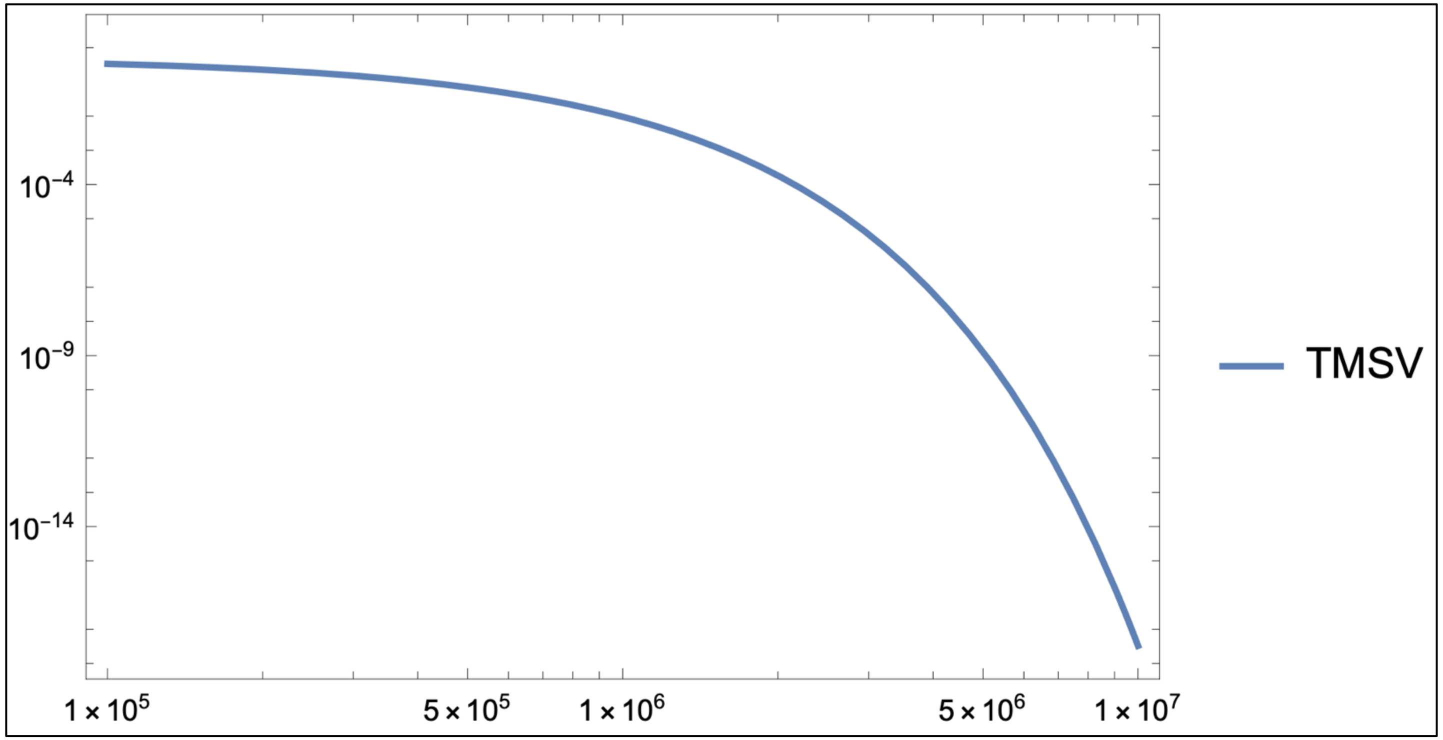
⋮

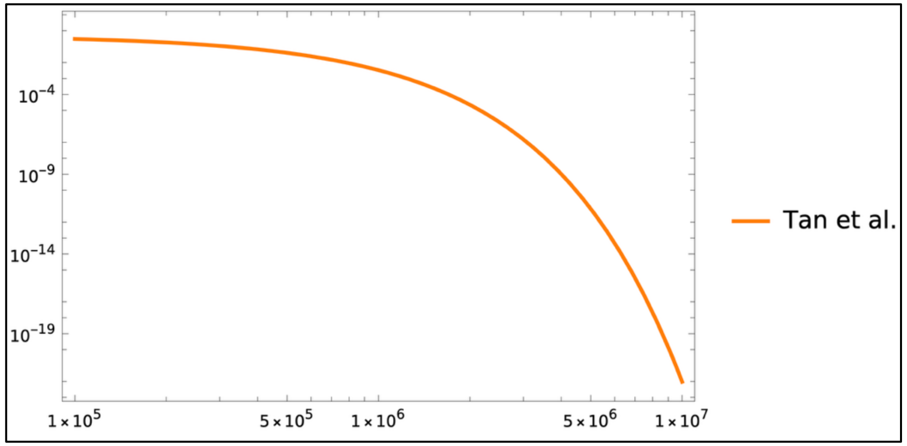
  - Unable to obtain an analytic expression of compact form (too long)
  - Fail to simplify (exceed the time limit 300s)

Reconstruction of Quantum illumination

- Calculate Bhattacharyya bound (for the TMSV transmitter)

Upper bounds on the target-detection error probabilities for TMSV





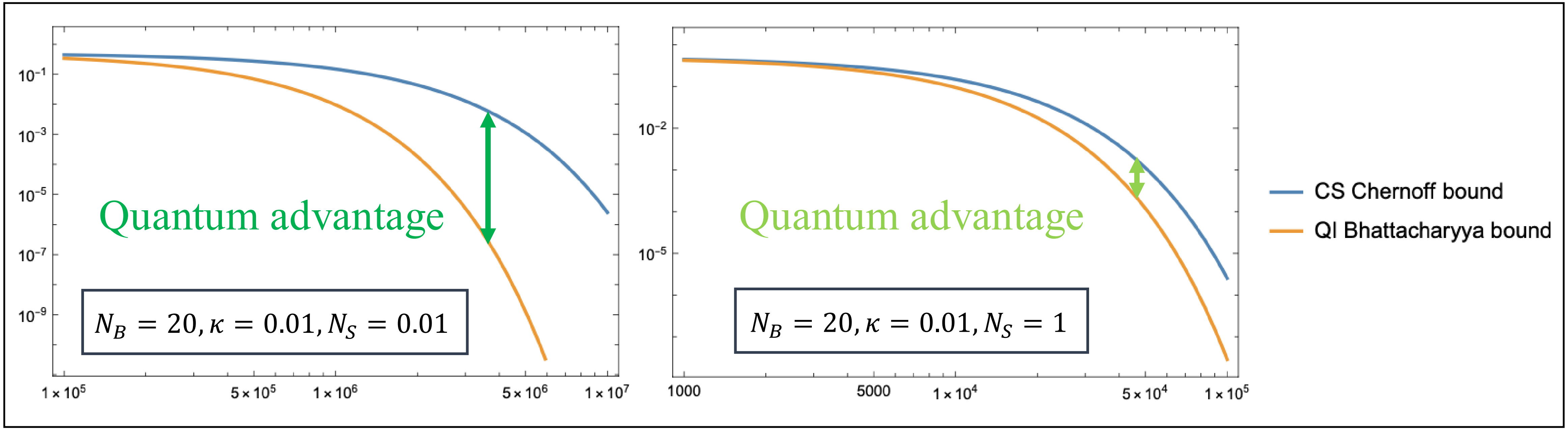
$$\Pr(e)_{\mathrm{QI}} \leq e^{-M\kappa N_S/N_B}/2$$

$$\kappa \ll 1, N_S \ll 1, N_B \gg 1$$

Results

- Quantum advantage becomes less and less important when increasing N_s

Upper bounds on the target-detection error probabilities for TMSV

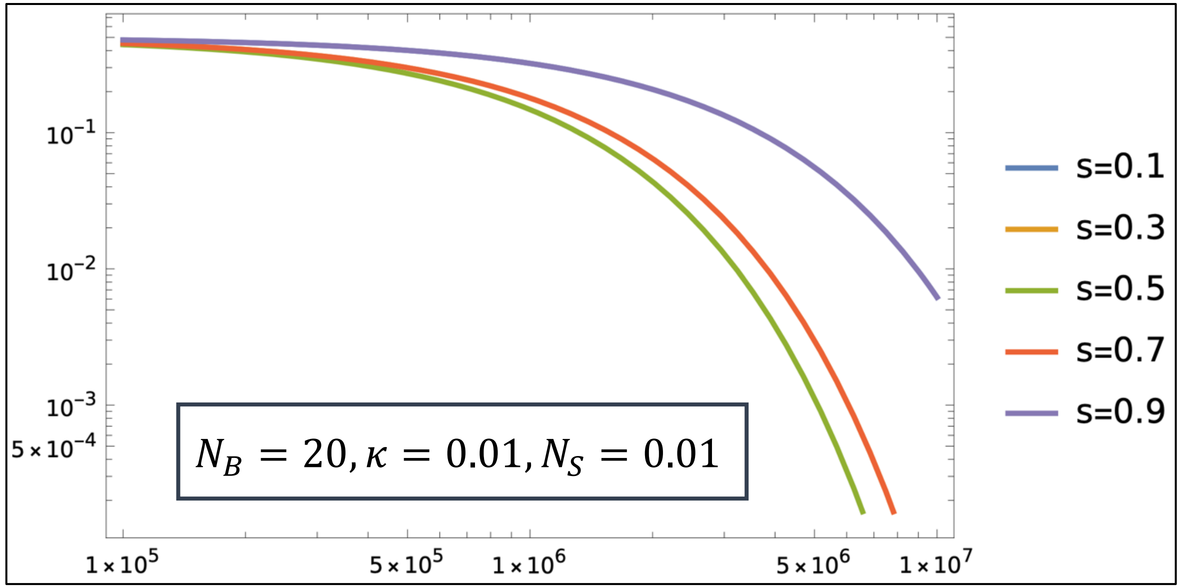


- Show the clear quantum advantage in the $N_S \ll 1$
- Quantum advantage is significantly depleted in the $N_S = 1$

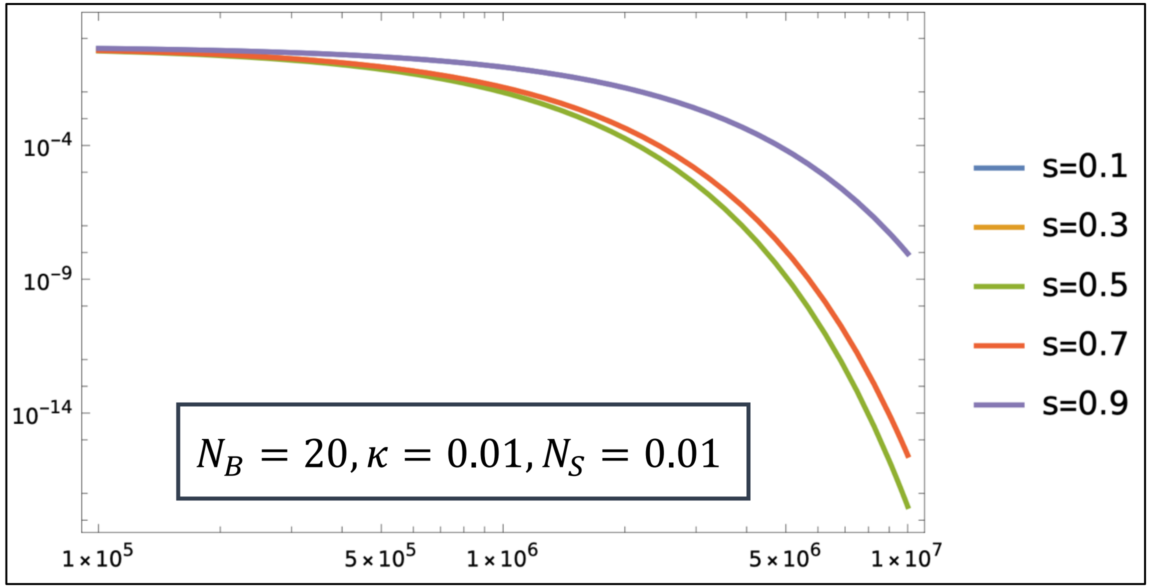
Results

- Target-detection error probabilities according to s (optimization over s)

Upper bounds on the P_e for the CI



Upper bounds on the P_e for the QI



- Symmetrical with respect to $s = 0.5$
- Lowest target-detection error probability at $s = 0.5$
- However, Chernoff bound ≠ Bhattacharyya bound in the QI (for the TMSV transmitter)

Discussion & Further research

- Reconstruction of Classical illumination
  - Find more simplified analytic expression by using approximation ($\kappa \ll 1, N_S \ll 1, N_B \gg 1$)
  - Check to have Chernoff bound at $s = 0.5$ by numerical method (implement Mathematica code)

- Reconstruction of Quantum illumination
  - Find analytic solution of compact form by using approximation ($\kappa \ll 1, N_S \ll 1, N_B \gg 1$)
  - Calculate quantum Chernoff bound by optimization over s

- Further research
  - Calculate target detection error probability for other Gaussian states
  - Calculate ROC for the coherent-state transmitter, TMSV transmitter
  - Implement illumination protocol using practical receiver (OPA, SS-SFG)

Reference & Acknowledgment

- Reference

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- Acknowledgment

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Thank you for your attention!